

GREEN FUNCTIONS OF THE SPECTRAL BALL AND SYMMETRIZED POLYDISK

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ABSTRACT. The Green function of the spectral ball is constant over the isospectral varieties, is never less than the pullback of its counterpart on the symmetrized polydisk, and is equal to it in the generic case where the pole is a cyclic (non-derogatory) matrix. When the pole is derogatory, the inequality is always strict, and the difference between the two functions depends on the order of nilpotence of the strictly upper triangular blocks that appear in the Jordan decomposition of the pole. In particular, the Green function of the spectral ball is not symmetric in its arguments. Additionally, some estimates are given for invariant functions in the symmetrized polydisc, e.g. (infinitesimal versions of) the Carathéodory distance and the Green function, that show that they are distinct in dimension greater or equal to 3.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let \mathcal{M}_n be the set of all $n \times n$ complex matrices. For $A \in \mathcal{M}_n$ denote by $sp(A)$ and $\rho(A) = \max_{\lambda \in sp(A)} |\lambda|$ the spectrum and the spectral radius of A , respectively. The notation $\|A\|$ will stand for an operator norm on the set of matrices (chosen once and for all).

The *spectral ball* Ω_n is the set

$$\Omega_n = \{A \in \mathcal{M}_n : \rho(A) < 1\}.$$

The characteristic polynomial of the matrix A is denoted

$$P_A(t) := \det(tI - A) =: t^n + \sum_{j=1}^n (-1)^j \sigma_j(A) t^{n-j},$$

where $I \in \mathcal{M}_n$ is the unit matrix. We define a map σ from \mathcal{M}_n to \mathbb{C}^n by $\sigma := (\sigma_1, \dots, \sigma_n)$. The *symmetrized polydisk* is $\mathbb{G}_n := \sigma(\Omega_n)$ is a bounded domain in \mathbb{C}^n , which is a complete hyperbolic domain, and hyperconvex (and thus taut).

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A matrix A is *cyclic* (or *non-derogatory*) if it admits a cyclic vector, we then write $A \in \mathcal{C}_n$. We say that A is derogatory when $A \notin \mathcal{C}_n$.

Definition 1.1. *The Green function with pole p in a domain Ω is given by*

$$g_\Omega(p, z) := \sup\{u(z) : u \in PSH_-(\Omega), u(w) \leq \log \|w - p\| + O(1)\}.$$

Let \mathbb{D} stand for the unit disk in \mathbb{C} .

Definition 1.2. *The Lempert function of a domain $D \subset \mathbb{C}^m$ is defined, for $z, w \in D$, as*

$$l_D(z, w) := \inf\{|\alpha| : \alpha \in \mathbb{D} \text{ and } \exists \varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(0) = z, \varphi(\alpha) = w\}.$$

Definition 1.3. *The Carathéodory (pseudo)distance for a domain $D \subset \mathbb{C}^m$ is defined, for $w, z \in D$, as*

$$c_D^*(z, w) := \sup\{|f(w)| : f \in \mathcal{O}(D, \mathbb{D}), f(z) = 0\}.$$

Immediate consequences of the definitions are that for any domain D in \mathbb{C}^n ,

$$\log c_D^*(z, w) \leq g_D(z, w) \leq \log l_D(z, w), \text{ and}$$

$$g_{\Omega_n}(V, M) \geq g_{\mathbb{G}_n}(\sigma(V), \sigma(M)).$$

One can prove that $\log l_{\Omega_n}(0, M) = g_{\Omega_n}(0, M) = \log \rho(M)$. This follows from Vesentini's theorem about the plurisubharmonicity of $\log \rho$ [13] and the facts that $\rho(\lambda A) = |\lambda| \rho(A)$, for $\lambda \in \mathbb{C}$ (see also [2, Theorem 3.4.7, p. 52] and [5]).

As is noted in [4], $\sigma(A) = \sigma(B)$ if and only if there is an entire curve contained in Ω_n going through A and B . It follows from Liouville's theorem for subharmonic functions that if $\sigma(M) = \sigma(M')$, then $g_{\Omega_n}(V, M) = g_{\Omega_n}(V, M')$. So $g_{\Omega_n}(V, M)$ only depends on $\sigma(M)$. One may wonder, then, whether for any V, M ,

$$g_{\Omega_n}(V, M) = g_{\mathbb{G}_n}(\sigma(V), \sigma(M))?$$

We will prove this only happens when $V \in \mathcal{C}_n$.

Let us proceed with some elementary reductions. For any $Q \in \mathcal{M}_n^{-1}$ (the set of invertible matrices), the map $M \mapsto Q^{-1}MQ$ is an automorphism of the spectral ball preserving the spectrum, so

$$g_{\Omega_n}(Q^{-1}VQ, M) = g_{\Omega_n}(V, QMQ^{-1}) = g_{\Omega_n}(V, M),$$

thus we may always assume that our pole matrix V is in Jordan form (or any other convenient reduction by conjugation).

For any $\lambda \in Sp(V)$, denote by V_λ the restriction of V to the stable subspace $\ker(V - \lambda I_n)^n$. Let $n(\lambda) := \dim(\ker(V - \lambda I_n)^n)$ (the size of the Jordan block corresponding to the eigenvalue λ) and $m(\lambda) := \min\{k :$

$(V_\lambda - \lambda I_{n(\lambda)})^k = 0$ the order of nilpotence of $V_\lambda - \lambda I_{n(\lambda)}$. Finally there exists $\lambda \in Sp(V)$ such that $m(\lambda) < n(\lambda)$ if and only if $V \notin \mathcal{C}_n$.

Theorem 1.4. *Let $V \in \Omega_n$.*

(1) *If $V \in \mathcal{C}_n$, then $g_{\Omega_n}(V, M) = g_{\mathbb{G}_n}(\sigma(V), \sigma(M))$.*

(2) *If $V \notin \mathcal{C}_n$, then there exists $X \in \mathcal{M}_n \setminus \{0\}$ such that*

$$(1.1) \quad g_{\Omega_n}(V, V + \zeta X) \geq m(\lambda) \log |\zeta| + O(1), \text{ while}$$

$$(1.2) \quad g_{\mathbb{G}_n}(\sigma(V), \sigma(V + \zeta X)) \leq n(\lambda) \log |\zeta| + O(1).$$

Proof. Part (1) follows from a theorem of Jarnicki and Pflug [6, Theorem 1], because the rank of the differential of σ at A is maximal precisely when $A \in \mathcal{C}_n$ [11]. Part (2) will be proved in sections 3 and 4 below. \square

The following result should be compared with [12, Theorem 1.3], which states that the continuity at A of $l_{\Omega_n}(\cdot, M)$, for any $M \in \Omega_n$, implies cyclicity of A (with the converse holding for $n \leq 3$, see [12, Proposition 1.4]).

Proposition 1.5. *Let $A, M \in \Omega_n$. The following properties are equivalent:*

(1) $g_{\Omega_n}(A, M) = g_{\mathbb{G}_n}(\sigma(A), \sigma(M))$.

(2) *The Green function g_{Ω_n} is continuous at (A, M) .*

(3) *The function $g_{\Omega_n}(\cdot, M)$ is continuous at A .*

An immediate corollary of Theorem 1.4 is that the function g_{Ω_n} is not symmetric in its arguments. Recall that both the Lempert function and the Carathéodory distance are symmetric (for all domains). Since $g_{\mathbb{G}_2} = \log l_{\mathbb{G}_2} = \log c_{\mathbb{G}_2}^*$ ([1], [3]) the Green function $g_{\mathbb{G}_2}$ is symmetric. We conjecture that $g_{\mathbb{G}_n}$ fails to be symmetric for $n \geq 3$.

Even though we cannot prove the above conjecture, we are able to get some estimates between (logarithm of) the Carathéodory distance and the Green function in the symmetrized polydisc, showing in particular that these two objects differ in \mathbb{G}_n , $n \geq 3$, which extends some of the results from [10]. We get this from facts about their infinitesimal versions. Recall that the *Carathéodory-Reiffen* and *Azukawa* pseudometrics in a domain $D \subset \mathbb{C}^n$ are respectively given by

$$\gamma_D(z, X) := \sup \{|f'(z) \cdot X| : f \in \mathcal{O}(D, \mathbb{D}), f(z) = 0\},$$

$$A_D(z, X) := \limsup_{\lambda \rightarrow 0} \frac{\exp g_D(z, z + \lambda X)}{|\lambda|}, \text{ for } z \in D, X \in \mathbb{C}^n.$$

Recall that one may replace 'lim sup' in the definition of the Azukawa metric above with 'lim' when D is a bounded hyperconvex domain (in

particular, when $D = \mathbb{G}_n$) – see e. g. [14]. We also make use of the fact that $\gamma_D(z, X) = \lim_{\lambda \rightarrow 0} \frac{c_D^*(z, z + \lambda X)}{|\lambda|}$ (see e. g. [5]).

Theorem 1.6. *For $n \geq 3$, $\gamma_{\mathbb{G}_n}(0; e_{n-1}) < A_{\mathbb{G}_n}(0; e_{n-1})$, and consequently $c_{\mathbb{G}_n}^*(0, te_{n-1}) < \exp g_{\mathbb{G}_n}(0, te_{n-1})$ for $|t|$ small enough.*

This follows from Proposition 5.4. The explicit estimates in Section 5 show that holomorphically invariant objects differ very much in \mathbb{G}_n , $n \geq 3$, in sharp contrast to the case $n = 2$.

2. PROOF OF PROPOSITION 1.5

That (2) implies (3) is clear.

Proof of (3) \Rightarrow (1).

Since the cyclic matrices are dense in Ω_n then there exist $A_j \in \mathcal{C}_n$ such that $A_j \rightarrow A$. By continuity of $g_{\Omega_n}(\cdot, M)$ at A , we get that $g_{\Omega_n}(A_j, M) \xrightarrow{j \rightarrow \infty} g_{\Omega_n}(A, M)$.

On the other hand, by Theorem 1.4(1), $g_{\Omega_n}(A_j, M)g_{\mathbb{G}_n}(\sigma(A_j), \sigma(M))$. By hyperconvexity of domain \mathbb{G}_n we have $g_{\mathbb{G}_n}(\sigma(A_j), \sigma(M)) \xrightarrow{j \rightarrow \infty} g_{\mathbb{G}_n}(\sigma(A), \sigma(M))$. This implies that $g_{\Omega_n}(A, M) = g_{\mathbb{G}_n}(\sigma(A), \sigma(M))$. Proof of (1) \Rightarrow (2).

Assume $g_{\Omega_n}(A, M) = g_{\mathbb{G}_n}(\sigma(A), \sigma(M))$.

Let $(A_j, M_j) \subset \Omega_n$ be such that $(A_j, M_j) \xrightarrow{j \rightarrow \infty} (A, M)$ and

$$\lim_{j \rightarrow \infty} g_{\Omega_n}(A_j, M_j) = a : \liminf_{(X, Y) \rightarrow (A, M)} g_{\Omega_n}(X, Y).$$

We have

$$g_{\Omega_n}(A_j, M_j) \geq g_{\mathbb{G}_n}(\sigma(A_j), \sigma(M_j)) \rightarrow g_{\mathbb{G}_n}(\sigma(A), \sigma(M)),$$

and hence $a \geq g_{\mathbb{G}_n}(\sigma(A), \sigma(M)) = g_{\Omega_n}(A, M)$. Then g_{Ω_n} is lower semi-continuous at (A, M) . Since g_{Ω_n} is upper semicontinuous [7], it is continuous at (A, M) .

3. PROOF OF THEOREM 1.4(2): THE NILPOTENT CASE

When we make the additional assumption that V is nilpotent, equivalently $Sp(V) = \{0\}$, we have $n(\lambda) = n$, $m(\lambda) = m := \min\{k : V^k = 0\}$, the order of nilpotence of V .

We begin by proving (1.1).

Lemma 3.1. *Let V, m be as in the hypotheses of Theorem 1.4 (2). Then $\log \rho(V+A) \leq \frac{1}{m} \log \|A\| + O(1)$, and as a consequence $g_{\Omega_n}(V, M) \geq m \log \rho(M)$, for any $M \in \Omega_n$.*

Proof. We assume that $V = (v_{ij})_{1 \leq i, j \leq n}$ is in Jordan form with the following notations. Let r stand for the rank of V . Write

$$F_0 := \{j : v_{ij} = 0 \text{ for } 1 \leq i \leq n\} := \{1 = b_1 < b_2 < \dots < b_{n-r}\}.$$

For all the other values of j , $v_{j-1, j} = 1$, $v_{ij} = 0$ for $i \neq j - 1$. We can choose the Jordan form so that $b_{l+1} - b_l$ is decreasing for $1 \leq l \leq n - r$, with the convention $b_{n-r+1} := n + 1$. With this choice of notation (and order), $m = b_2 - b_1$.

Now we must study the homogeneity of the functions $\sigma_i(V + A)$ in terms of the entries of A . This is Lemma 4.2 from [12].

Lemma 3.2. *Let $d_i := 1 + \#(F_0 \cap [(n - i + 2)..n])$. The lowest order terms of $\sigma_i(V + A)$ are of degree d_i (in the entries of A).*

Then the eigenvalues $(\lambda_1, \dots, \lambda_n)$ of $V + A$ satisfy the following equations:

$$s_i(\lambda_1, \dots, \lambda_n) = \sigma_i(V + A), 1 \leq i \leq n,$$

where $s_i(\lambda_1, \dots, \lambda_n)$ stands for the elementary symmetric function of degree i .

Lemma 3.3. *$md_i \geq i$, for $1 \leq i \leq n$.*

Then we set $\lambda' := \lambda \|A\|^{-1/m}$, and we have the new equations (for $A \neq 0$)

$$s_i(\lambda'_1, \dots, \lambda'_n) = \sigma_i(V + A) \|A\|^{-i/m}, 1 \leq i \leq n,$$

and by the Lemma the right hand sides are bounded functions of A near 0. Since a polynomial of the form $X^n + \sum_j \alpha_j X^j$ where $|\alpha_j| \leq C$ has all its roots in a disk of radius $Cn^{1/n}$ about the origin, all the solutions of those equations are bounded by a constant (which depends on V), thus $\lambda = O(\|A\|^{1/m})$. Taking logarithms, we find the desired estimate on u .

Proof of Lemma 3.3.

Suppose that $b_l \leq i < b_{l+1}$. Then $d_i = l$, so it will be enough to prove that $ml \geq b_{l+1} - 1$, for any $l \leq n - r$. But, by our hypothesis of decrease of the $b_{j+1} - b_j$,

$$b_{l+1} - 1 = \sum_{j=1}^l b_{j+1} - b_j \leq \sum_{j=1}^l b_2 - b_1 = lm.$$

□

□

Remark.

The bound in Lemma 3.1 is optimal. Indeed, recall that $m = b_2 - b_1 = b_2 - 1$. Let $X := (x_{ij})$ where $x_{m1} = 1, x_{ij} = 0$ otherwise. Then

$$P_{V+\zeta X}(t) = (t^m - \zeta)t^{n-m},$$

so $\rho(V + \zeta X) = |\zeta|^{1/m}$. The map $\psi(\zeta) = V + \zeta X$ sends \mathbb{D} to Ω_n , so $l_{\Omega_n}(V, V + \zeta X) \leq |\zeta|$, so

$$g_{\Omega_n}(V, V + \zeta X) \leq \log l_{\Omega_n}(V, V + \zeta X) \leq \log |\zeta| = m \log \rho(V + \zeta X).$$

To prove (1.2), choose a matrix X with $\sigma_i(X) = 0$ for $i \leq n-1$, $\sigma_n(X) = (-1)^{n-1}$ (the spectrum is then made up of all the n -th roots of unity). Then

$$\sigma(\zeta X) = \zeta^n(0, \dots, 0, \sigma_n(X)).$$

Therefore $g_{\mathbb{G}^n}(0, \sigma(\zeta X)) \leq n \log |\zeta| + O(1)$. \square

To see more general cases of matrices X where the Green function of the spectral ball is strictly above the pull back $g_{\mathbb{G}^n} \circ \sigma$, take X such that its characteristic polynomial verifies $\sigma_i(X) = 0$ for $i \leq m$, and that its eigenvalues are all distinct and nonzero. This is always possible, since $m \leq n-1$. Then $g_{\mathbb{G}^n}(0, \sigma(\zeta X)) \leq (m+1) \log |\zeta| + O(1) \leq (m+1) \log \rho(V + \zeta X) + O(1) < g_{\Omega_n}(0, \zeta X)$ for ζ small enough.

4. PROOF OF THE THEOREM: GENERAL CASE

Let λ_0 be an eigenvalue such that $m(\lambda_0) := m_0 < n(\lambda_0) := n_0$. By applying the automorphism $M \mapsto (\lambda_0 I_n - M)(I_n - \lambda_0 M)^{-1}$, we may reduce ourselves to the case $\lambda_0 = 0$, and we may assume further that

$$V = \begin{pmatrix} V_0 & 0 \\ 0 & V_1 \end{pmatrix},$$

where $V_0 \in \mathcal{M}_{n_0}$ is in Jordan form.

Lemma 4.1. *There exist a neighborhood \mathcal{U} of $\sigma(V)$ in \mathbb{G}_n and σ^0 a holomorphic map from $\sigma^{-1}(\mathcal{U})$ to \mathbb{C}^{n_0} such that*

$$X^{n_0} + \sum_{j=1}^{n_0} (-1)^j \sigma_j^0(M) X^{n_0-j} := P_M^0(X) = (X - \lambda_1) \cdots (X - \lambda_{n_0}),$$

where $\{\lambda_1, \dots, \lambda_{n_0}\}$ are the smallest n_0 eigenvalues of M (in modulus).

Proof. This fact relies on the holomorphic dependency of a subset of the roots of a polynomial in a neighborhood of a multiple root, in the spirit of the Weierstrass Preparation Theorem.

In more detail: for $s = (s_1, \dots, s_n) \in \mathbb{G}_n$, let $P_s(X) = X^n + \sum_{j=1}^n (-1)^j s_j X^{n-j}$. There exists $\delta > 0$ such that the open set

$$\mathcal{U}_\delta := \{s \in \mathbb{G}_n : \#(P_s^{-1}\{0\} \cap D(0, \delta)) = n_0, P_s^{-1}\{0\} \cap \partial D(0, \delta) = \emptyset\},$$

where the zeroes are counted with multiplicities, contains $\sigma(V)$. On $\sigma^{-1}(\mathcal{U}_\delta)$, the formulas

$$\Sigma_k(M) := \frac{1}{2\pi i} \int_{\partial D(0, \delta)} \zeta^k \frac{(P^0)'_M(\zeta)}{P_M(\zeta)} d\zeta$$

give holomorphic functions which are equal to $\lambda_1^k + \dots + \lambda_{n_0}^k$, and the elementary symmetric functions of that subset of eigenvalues can be algebraically deduced from those. \square

Notice that the above lemma gives a holomorphically varying factorization of the characteristic polynomial of M : $P_M(X) = P_M^0(X)P_M^1(X)$, and a holomorphically varying splitting of the space \mathbb{C}^n ,

$$\mathbb{C}^n = \ker P_M^0(M) \oplus \ker P_M^1(M) =: U^0 \oplus U^1.$$

Then $P_M^0 = P_{M|_{U^0}}$ and $\rho^0(M) := \rho(M|_{U^0})$ is the largest modulus of the eigenvalues of M contained in $D(0, \delta)$. So $u(M) := \log \rho^0(M)$ defines a plurisubharmonic function in a neighborhood of V .

We follow the scheme of proof of the special case.

Since $g_{\mathbb{G}_n}(\sigma(V), \cdot) = -\infty$ precisely at the point $\sigma(V)$ and $g_{\Omega_n}(V, M) \geq g_{\mathbb{G}_n}(\sigma(V), \sigma(M))$, we can pick an $\varepsilon_0 > 0$ such that

$$\mathcal{U}_0 := \sigma(\{g_{\Omega_n}(V, \cdot) < \log \varepsilon_0\}) \subset \mathcal{U}_\delta.$$

Therefore

$$\sigma^{-1}(\mathcal{U}_0) = \{g_{\Omega_n}(V, \cdot) < \log \varepsilon_0\} \subset \sigma^{-1}(\mathcal{U}_\delta)$$

(recall that $g_{\Omega_n}(V, \cdot)$ is constant on the fibers of σ). It is a standard fact that then

$$g_{\sigma^{-1}(\mathcal{U}_0)}(V, \cdot) = g_{\Omega_n}(V, \cdot) - \log \varepsilon_0.$$

To compare this local Green function with our function u , it is enough to estimate u near the pole V .

Lemma 4.2. *There exists a neighborhood \mathcal{V} of 0 in \mathcal{M}_n such that for any $A \in \mathcal{V}$, $u(V+A) \leq \frac{1}{m_0} \log \|A\| + O(1)$, and therefore $g_{\sigma^{-1}(\mathcal{U}_0)}(V, M) \geq m_0 u(M)$.*

This will conclude the proof, since we can find a matrix X (work as before, but only on the upper left block) such that $g_{\mathbb{G}_n}(\sigma(V), \sigma(V + \zeta X)) \leq n_0 u(V + \zeta X) + O(1)$. \square

Proof of Lemma 4.2

For A small enough, $\ker P_{V+A}^0(V+A)$ (respectively $\ker P_{V+A}^1(V+A)$) is close enough to $\ker P_V^0(V) = \mathbb{C}^{n_0} \times \{0\}$ (respectively to $\ker P_V^1(V) = \{0\} \times \mathbb{C}^{n-n_0}$) so that the projections π_j from \mathbb{C}^n to $\ker P_V^j(V)$ with kernel equal to $\ker P_{V+A}^{1-j}(V+A)$ ($j = 0, 1$) induce bijections from $\ker P_{V+A}^j(V+A)$ onto $\ker P_V^j(V)$.

Let P be the matrix of the bijective endomorphism defined by $\pi_0|_{\ker P_{V+A}^0(V+A)} + \pi_1|_{\ker P_{V+A}^1(V+A)}$. Then

$$PMP^{-1} = \begin{pmatrix} M_0 & 0 \\ 0 & M_1 \end{pmatrix},$$

for some $M_0 \in \mathcal{M}_{n_0}$ and $M_1 \in \mathcal{M}_{n-n_0}$. We have seen that $\{\lambda_1, \dots, \lambda_{n_0}\} = SpM_0$, and one can check that $P = I_n + O(\|A\|)$, so that $M_0 = V_0 + O(\|A\|)$. Applying the proof in Section 3, $\lambda_j = O(\|M_0 - V_0\|^{1/m_0}) = O(\|A\|^{1/m_0})$, for $1 \leq j \leq n_0$. The estimate follows easily. \square

5. ESTIMATES BETWEEN THE GREEN FUNCTION AND THE CARATHÉODORY DISTANCE IN \mathbb{G}_n , $n \geq 3$

This part of the paper may be seen as a continuation and extension of the results from [10]. Recall [5] that for any $k \in \mathbb{Z}_+^*$,

$$\gamma_D^{(k)}(z, X) := \sup \left\{ \limsup_{\lambda \rightarrow 0} \frac{|f(z + \lambda X)|^{1/k}}{|\lambda|}, f \in \mathcal{O}(D, \mathbb{D}), \text{ord}_z f \geq k \right\},$$

and that $\kappa_D(z, X) \geq A_D(z, X) \geq \gamma_D^{(k)}(z, X) \geq \gamma_D(z, X)$.

The definitions and basic properties of some additional infinitesimal functions used below (Kobayashi-Royden metric κ_D and Kobayashi-Buseman metric $\hat{\kappa}_D$) may be found in [10] or [5], with identical notations.

Proposition 5.1. *For any $n \geq 2$ the following inequalities hold*

$$\kappa_{\mathbb{G}_n}(0; e_{n-1}) \geq A_{\mathbb{G}_n}(0; e_{n-1}) \geq \gamma_{\mathbb{G}_n}^{(n-1)}(0; e_{n-1}) \geq \sqrt[n-1]{(n-1)/n}.$$

Proof. We only need to prove the last inequality.

Recall that $\mathbb{G}_n = \pi(\mathbb{D}^n)$, where, with the notation of Section 3 for the elementary symmetric functions,

$$\pi_j(\lambda_1, \dots, \lambda_n) : (s_1(\lambda_1, \dots, \lambda_n), s_2(\lambda_1, \dots, \lambda_n), \dots, s_n(\lambda_1, \dots, \lambda_n)).$$

Consider the function $f(\lambda_1, \dots, \lambda_n) := (\lambda_1^l + \dots + \lambda_n^l)/n$, $\lambda_j \in \mathbb{D}$. We may treat f as a function from $\mathcal{O}(\mathbb{G}_n, \mathbb{D})$. Recall that it is a polynomial. To get the lower estimate for the Azukawa metric at 0 in direction e_{n-1} we want the function f to be the function of multiplicity at 0 at least k and we want the power at z_{n-1} to be equal to k . Therefore, we want l to be $k(n-1)$. Then it follows from the Waring formula that

the absolute value of the coefficient at z_{n-1}^k is equal to $(n-1)/n$. The function f (as a function on \mathbb{G}_n) has only powers with degree not less than k iff $k \leq n-1$. Therefore, we fix below $k = n-1$. We get the following lower estimate

$$\kappa_{\mathbb{G}_n}(0; e_{n-1}) \geq A_{\mathbb{G}_n}(0; e_{n-1}) \geq \gamma_{\mathbb{G}_n}^{(n-1)}(0; e_{n-1}) \geq \sqrt[n-1]{(n-1)/n}.$$

□

Remark 5.2. *The estimate above is better (especially asymptotically) than the general one from [9] (which is $(n-1)/n$).*

Remark 5.3. *Unfortunately, because of the form of the function f above we do not have the lower estimate $\hat{\gamma}_{\mathbb{G}_n}^{(n-1)}(0; e_{n-1})$ with the same constant (with the methods from [10]). Consequently, we do not get the strict inequality between $\gamma_{\mathbb{G}_n}(0; e_{n-1})$ and $\hat{\kappa}_{\mathbb{G}_n}(0; e_{n-1})$, $n \geq 4$.*

We may also improve the upper estimate for the Carathéodory-Reiffen pseudometric so that we shall get the inequality between the Azukawa and Carathéodory-Reiffen metric on the symmetrized polydisc (and therefore also between the Green function and the Carathéodory pseudodistance).

Proposition 5.4. *Let $n \geq 3$. Then the following inequality holds*

$$\gamma_{\mathbb{G}_n}(0; e_{n-1}) \leq \frac{1 + (n/(n-2))^{n-1}}{n/(n-2) + (n/(n-2))^{n-1}}.$$

In particular, for $n \geq 4$

$$\gamma_{\mathbb{G}_n}(0; e_{n-1}) < \gamma_{\mathbb{G}_n}^{(n-1)}(0; e_{n-1}) \leq A_{\mathbb{G}_n}(0; e_{n-1}).$$

Remark 5.5. *Note that the numbers $\gamma_{\mathbb{G}_n}(0; e_{n-1})$ and $A_{\mathbb{G}_n}(0; e_{n-1})$ differ very distinctly asymptotically. It is elementary to see that*

$$\liminf_{n \rightarrow \infty} (n(1 - \gamma_{\mathbb{G}_n}(0; e_{n-1}))) \geq 2/(1 + e^2)$$

whereas $\lim_{n \rightarrow \infty} n(1 - A_{\mathbb{G}_n}(0; e_{n-1})) = 0$.

Proof of Theorem 1.6. For $n \geq 4$, this is Proposition 5.4. It follows from [10, Proposition 5] that $\gamma_{\mathbb{G}_3}(0; e_2) < A_{\mathbb{G}_3}(0; e_2)$. □

Proof of Proposition 5.4. From [10, Proposition 3], for any $n \geq 3$ we have the equality $\gamma_{\mathbb{G}_n}(0; e_{n-1}) = 1/M_n$, with

$$M_n := \inf_{a \in \mathbb{C}^{\mathcal{P}_n}} \max\{|z_{n-1} + \sum_{\alpha \in \mathcal{P}_n} a_\alpha z^\alpha| : z \in \partial \mathbb{G}_n\},$$

where \mathcal{P}_n stands for the set of all $(n-2)$ -tuples of non-negative integers α such that $\alpha_1 + 2\alpha_2 + \dots + (n-2)\alpha_{n-2} = n-1$. We proceed as in

that paper; however, much more effort is required to find appropriate polynomials.

Notice that the coefficients of monic polynomials with all zeros lying on the unit circle deliver elements $z \in \partial\mathbb{G}_n$, with the notation

$$p(\lambda) = \lambda^n + \sum_{j=1}^n (-1)^j z_j \lambda^{n-j}.$$

We shall consider two kinds of such polynomials, both with the property that $z_j = 0$, $2 \leq j \leq n-2$. Restricting to this subclass implies

$$(5.1) \quad M_n \geq \inf_{a_{(n-1,0,\dots,0)} \in \mathbb{C}} \max\{|z_{n-1} + a_{(n-1,0,\dots,0)} z_1^{n-1}| : (z_1, 0, \dots, 0, z_{n-1}, z_n) \in \partial\mathbb{G}_n\}.$$

From now on we write $a = a_{(n-1,0,\dots,0)}$.

The first polynomial is $(\lambda^{n-1} - 1)(\lambda - 1)$, which gives that $(1, 0, \dots, 0, (-1)^n, (-1)^n) \in \partial\mathbb{G}_n$. To find another good polynomial we need more subtle methods. Recall that a polynomial $p(\lambda) = \sum_{j=0}^n a_j \lambda^j$ with $a_n \neq 0$ is called *self-inversive* if $a_{n-j} = \epsilon \bar{a}_j$, $j = 0, \dots, n$ for some $|\epsilon| = 1$.

Lemma 5.6. *For all $n \in \mathbb{Z}$, $n \geq 3$, all $t \in I_n := [(-1)^n - \frac{2}{n-2}, (-1)^n + \frac{2}{n-2}]$, the self-inversive polynomial*

$$p_{n,t}(\lambda) := \lambda^n + (-1)^{n-1} t \lambda^{n-1} + t \lambda + (-1)^{n-1}$$

has all its roots lying on the unit circle.

Then the point $((-1)^n t, 0, \dots, 0, (-1)^{n-1} t, -1)$ belongs to $\partial\mathbb{G}_n$.

From (5.1) we see that

$$M_n \geq \inf_{a \in \mathbb{C}} \max_{t \in I_n} \left(\max(|(-1)^n + a 1^{n-1}|, |(-1)^{n-1} t + a t^{n-1}|) \right),$$

therefore for any $t \in I_n$,

$$M_n \geq M_n^t := \inf_{a \in \mathbb{C}} \left(\max(|(-1)^n + a|, |(-1)^{n-1} t + a t^{n-1}|) \right).$$

Since the function over which the last infimum is taken is coercive, there exists an $a(t) \in \mathbb{C}$ such that $M_n^t = \max(|(-1)^n + a(t)|, |(-1)^{n-1} t + a(t) t^{n-1}|)$. Therefore

$$(|t|^n + 1) M_n^t \geq |(-1)^n t^{n-1} + a(t) t^{n-1}| + |(-1)^n t - a(t) t^{n-1}| \geq |t^{n-1} + t|,$$

and consequently, $M_n \geq |t^{n-1} + t| / (1 + |t|^{n-1})$, for any $t \in I_n$.

Taking $t = (-1)^{n-1} (1 + 2/(n-2))$, we have $\gamma_n(0; e_{n-1}) \leq \frac{1 + (n/(n-2))^{n-1}}{n/(n-2) + (n/(n-2))^{n-1}}$.

□

Proof of Lemma 5.6.

We may write that

$$\begin{aligned}
 p_{n,t}(\lambda) &= (\lambda + 1)(\lambda^{n-1} - (1 + (-1)^n t)\lambda^{n-2} + (1 + (-1)^n t)\lambda^{n-3} + \dots \\
 &\quad \dots + (-1)^{n-2}(1 + (-1)^n t)\lambda + (-1)^{n-1}) \\
 &=: (\lambda + 1)q_{n,t}(\lambda).
 \end{aligned}$$

Since $q_{n,t}$ is a self-inversive polynomial we may make use of Theorem 1 of [8] (take $B = c = -d = 1$) and we conclude that if $2 \geq (n - 2)|1 + (-1)^n t|$ then all zeros of $q_{n,t}$ (and consequently all the zeros of $p_{n,t}$) lie on the unit circle as claimed. \square

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